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The Equivalence Principle According to Mach

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A prerelativistic Machian theory of gravitation in a relative configuration space of the type developed in Barbour and Bertotti (1977) is proposed, which fulfils the principle of equivalence in a natural way. This is accomplished by assuming that the basic interactions with which the dynamical Lagrangian is constructed are three-body and velocity dependent. Gravity arises between two bodies when other masses move—in particular when the universe expands (or contracts). The properties and physical consequences of this theory are very similar to the previous one; in particular the two-body problem has a small post-Newtonian correction leading to an advance of the periastron, and to the determination of the velocity of expansion of the universe. We find that the motion of test particles introduces naturally into the theory the restricted covariance group, in which any space transformation that preserves simultaneity is allowed. This permits us to define an inertial frame of reference, and to obtain the analog of the equation of geodesic deviation. Finally, we discuss the effect of the anisotropy of the universe on the mass.

1. INTRODUCTION

J. B. Barbour (1974a, b; 1975) and Barbour with Bertotti (1977) (referred to as BB) have shown that Leibniz's (Leibniz and Clark, 1956), Berkeley's (1710, 1721)—and especially Mach's (1960)—ideas about the theory of motion (if followed consistently and faithfully) suggest an approach radically

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different from both Newtonian and relativistic dynamics, which has never been fully explored. In this approach the laws of motion acquire their usual form only in a stable cosmological environment, and with a very large distribution of masses at great distances; if only a few bodies were present, the basic physical laws would appear very different (and have been called "*protophysics*"). In the previous paper BB have shown the great power and richness of two precise mathematical principles, which we believe are the most faithful realization so far of Mach's ideas about motion in a prerelativistic framework: the relative configuration space (RCS), and the arbitrariness of the time variable. Essentially, the whole classical (i.e., nonrelativistic) gravitation physics was recovered, including the value of the gravitational constant and the advance of the periastron in the Kepler problem (which was very well known in the last century). This was achieved using very simple techniques and a Euclidean three-dimensional space. It seems, however, that the strong principle of equivalence, Lorentz invariance, and retardation lie at a much deeper level—if indeed they are encompassed at all in this framework (of course, if this is not the case, our work will have no practical application and should be regarded only as a stimulating exploration of historical possibilities).

In this paper, we present a much better formulation of the theory developed in BB which overcomes a serious objection, namely, the weak equivalence principle is imposed there a priori through an artificially constructed Lagrangian. In the present paper it is achieved in a very simple way, while still retaining all the better features of BB. This is accomplished by assuming that the elementary interactions with which the dynamical Lagrangian is constructed, are *velocity-dependent, three-body interactions*: Then two bodies attract each other via the rest of the universe. Their relative velocity with respect to the universe, which is essentially its speed of expansion, determines the strength of gravity. The formalism is very similar to BB, and is developed in Sections 3 and 4; in the Conclusion we discuss in detail the weak principle of equivalence in this theory.

2. SIMPLICITY IS NOT OBVIOUS

The reason why BB failed to incorporate the equivalence principle was the use of elementary two-body interactions in the construction of the Lagrangian. This was, of course, suggested by ordinary physics, but on further thought it is not necessarily the simplest choice from a Machian point of view. We expect, in fact, that a body A and a body B interact only in the presence of other bodies. This is well borne out by our important conclusion (see BB, Section 4) that a problem with only two bodies has no dynamical content: Their successive configurations are described by a single variable, the distance, which is an arbitrary function of the arbitrary time. Our old action, which yielded a well-defined expression (but one devoid of dynamical content) even for the two-body problem, was therefore redundant.

We want to construct a Machian theory based upon elementary three-body interactions. In this view, a two-body interaction can be described symbolically as follows:

$$\text{interaction between } A \text{ and } B = \sum_{C} (\text{interactions between } A, B, C)$$

where the summation is taken over all the other bodies in the universe. We also choose to realize the time invariance by the generalized "geodesic principle": The Lagrangian has the form

$$\{\text{quadratic form in the time derivatives of the observables}\}^{1/2} \tag{2.1}$$

Taking, as before, the Euclidean distances r_{ij} between the points i and j as the dynamical variables, we consider three-body terms proportional to

$$\{\dot{r}_{ij}^2 + \dot{r}_{jk}^2 + \dot{r}_{ki}^2\}$$

weighted with the product of their (positive) masses, $m_i m_j m_k$. We must also add a factor that makes the interaction weaker when the three bodies are far apart; the simplest choice is the Euclidean area S_{ijk} of the triangle they form. This leads to the Lagrangian

$$\mathscr{L} = \left[\sum_{i<j<k} m_i m_j m_k \frac{(\dot{r}_{ij}^2 + \dot{r}_{jk}^2 + \dot{r}_{kl}^2)}{S_{ijk}}\right]^{1/2} \tag{2.2}$$

The dot indicates a derivative with respect to the arbitrary time parameter. No other factor is needed to describe gravitation.

Of course, other choices of a Lagrangian are possible—such as $\dot{r}_{ij}\dot{r}_{jk}$, or a four-body interaction. The adoption of a Euclidean infinite background space is a weak point from a Machian point of view, and leads to an awkward cosmology, as explained in BB (Section 5). The latter difficulty could be remedied by adopting as a basic geometry a three-dimensional hypersphere embedded in a four-dimensional Euclidean space; one obtains in this way very similar results. At the present rudimentary and nonrelativistic level, however, it seems pointless to consider refinements and completeness. A simple spherical shell of radius R and mass M is thus adopted as the cosmology (as in BB), and the motion computed in the *cosmic limit*: $m/M \to 0$, $r/R \to 0$; where r is the distance from the center of the shell and m is a typical local mass.

It is noteworthy that, as expected, the problem of one and two bodies has no dynamical content; the problem of three bodies is simpler than in BB because it does not depend on their masses. Denoting by S the area of their triangle, they move according to the Lagrangian

$$\mathscr{L} = \left(\frac{\dot{r}_{12}^2 + \dot{r}_{23}^2 + \dot{r}_{31}^2}{S}\right)^{1/2}$$

the trajectory is a ray in the space (r_{12}, r_{23}, r_{31}), with index of refraction $n = 1/S^{1/2}$. It can be shown that a boundary $S = 0$ is approached normally; the ray is reflected back into the allowed region $S > 0$ with a cusped trajectory. Note that the mass of a body has a physical significance only if there are at least four bodies in the universe.

3. LOCAL PHYSICS

Before going into the actual calculations, it is helpful to summarize their physical meaning. The triple summation appearing in (2.2) splits into four groups of terms, according as a point belongs to the cosmological shell, or to the local group at or near its center. In estimating the order of magnitude we have assumed that all the velocities are of the same order as the velocity of the expansion of the universe, $\dot{R}$; thus, the only smallness parameter is m/M, while "Mach's ratio"

$$\chi = mR/rM \tag{3.1}$$

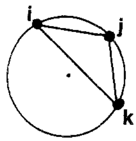
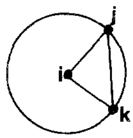
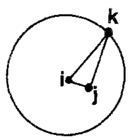
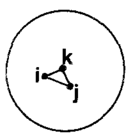
is considered finite. In the actual applications, however, only $\chi \ll 1$ will be considered. Referring now to Table I, in the cosmic limit $m/M \to 0$ the value of the cosmological term (i) is dynamically irrelevant, and the local Lagrangian is just proportional to $R/\dot{R}\{(\text{ii}) + (\text{iii}) + (\text{iv})\}$, thereby losing the time invariance.

In the actual calculations of the Lagrangian, full use is made of the uniformity of mass distribution on the shell. Possible anisotropy effects can be described by adding more local masses. In accordance with the cosmic limit, the quantity $\dot{r}_{ij}$ for a local point $\mathbf{r}_i$ coupled with a shell point $R\hat{\mathbf{r}}$ is taken to be $\dot{R} - \hat{\mathbf{r}}\cdot\dot{\mathbf{r}}_i$.

The resulting Lagrangian for a system of point masses near the center of the cosmological shell, and using an arbitrary time parameter, is

$$\begin{aligned}\mathscr{L}_L = {}& \frac{1}{6R\dot{R}}\sum_i m_i|\dot{\mathbf{r}}_i|^2 + \frac{\dot{R}}{M}\sum_{i<j}\frac{m_i m_j}{r_{ij}} + \frac{1}{2M\dot{R}}\sum_{i<j}\frac{m_i m_j}{r_{ij}}\dot{r}_{ij}^2 \\ & + \frac{1}{8M\dot{R}}\sum_{i<j}\frac{m_i m_j}{r_{ij}}[|\dot{\mathbf{r}}_i|^2 + |\dot{\mathbf{r}}_j|^2 + (\hat{\mathbf{r}}_{ij}\cdot\dot{\mathbf{r}}_i)^2 + (\hat{\mathbf{r}}_{ij}\cdot\dot{\mathbf{r}}_j)^2] \\ & + \frac{R}{2\pi M^2\dot{R}}\sum_{i<j<k}\frac{m_i m_j m_k}{S_{ijk}}(\dot{r}_{ij}^2 + \dot{r}_{jk}^2 + \dot{r}_{ki}^2)\end{aligned} \tag{3.2}$$

TABLE I. Grouping interaction terms: local dynamics from protophysics.

	Group	Order of Magnitude	Significance
(i)		$\frac{M^3\dot{R}^2}{R^2} \sim 1$	Cosmological term
(ii)		$\frac{M^2m\dot{R}^2}{R^2} \sim \frac{m}{M}$	Kinetic energy of the point i
(iii)		$\frac{M}{r}\frac{m^2\dot{R}^2}{R} \sim \frac{m}{M}\cdot\frac{mR}{Mr}$	Interaction energy between points i and j; perihelion perturbing terms
(iv)		$\frac{m^3\dot{R}^2}{r^2} \sim \frac{m}{M}\cdot\left(\frac{m}{r}\frac{R}{M}\right)^2$	Three-body interaction of a new kind

As yet, R is an arbitrary function of time. To recover a Lagrangian satisfying the first principle of dynamics (Newton's First Law), choose

$$\dot{R} = 1/3R \tag{3.3}$$

which thus defines the local time t as a function of the radius of the universe. Note that just as in the steady-state theory (Bondi, 1952), the homogeneous and isotropic cosmology is unique; here, however, $q = -\ddot{R}R/\dot{R}^2 = 1$. This is the main observational parameter distinguishing RCS theories of this kind; BB produced $q = 3/2$ [see BB (6.11)]. When this local time is adopted, $\mathscr{L}_L$ reads

$$\begin{aligned}\mathscr{L}_L = {} & \frac{1}{2}\sum_i m_i|\dot{\mathbf{r}}_i|^2 + \frac{3R\dot{R}^2}{M}\sum_{i<j}\frac{m_i m_j}{r_{ij}} + \frac{3R}{2M}\sum_{i<j}\frac{m_i m_j}{r_{ij}}\dot{r}_{ij}^2 \\ & + \frac{3R}{8M}\sum_{i<j}\frac{m_i m_j}{r_{ij}}[|\dot{\mathbf{r}}_i|^2 + |\dot{\mathbf{r}}_j|^2 + (\hat{\mathbf{r}}_{ij}\cdot\dot{\mathbf{r}}_i)^2 + (\hat{\mathbf{r}}_{ij}\cdot\dot{\mathbf{r}}_j)^2] \\ & + \frac{3R^2}{2\pi M^2}\sum_{i<j<k} m_i m_j m_k \frac{(\dot{r}_{ij}^2 + \dot{r}_{jk}^2 + \dot{r}_{ki}^2)}{S_{ijk}}\end{aligned} \tag{3.4}$$

The ordinary two-body gravitational interaction (the second term), arises quite naturally from the triangle defined by a point on the shell, and two near the center; its area will be proportional to r_{ij}. The correct value of the gravitational constant

$$G = 3R\dot{R}^2/M \tag{3.5}$$

arises because the fundamental interaction is velocity dependent and dominated by the expansion of the universe. In BB and in Barbour (1975) the "internal motion" mechanism was considered in order to generate gravity (Section 10); ours could be called an "external motion" mechanism and is more satisfactory because it relies on a well-defined physical velocity. In either case, it is interesting to note that a direct Machian approach to the problem of inertia using only relative distances produces gravity-type forces rather naturally as a by-product of the solution of the problem of inertia.

4. CELESTIAL MECHANICS

In the slow-motion approximation ($|\dot{\mathbf{r}}|^2/\dot{R}^2 = O(\chi) \ll 1$), the four types of terms in (3.4) have the following magnitudes:

$$m\dot{r}^2$$

$$\frac{Gm^2}{r} = m\dot{R}^2 \cdot \frac{Rm}{Mr} = m\dot{R}^2\chi = O(m\dot{r}^2)$$

$$\frac{Gm^2}{r} \cdot \frac{\dot{r}^2}{\dot{R}^2} = O(m\dot{r}^2\chi)$$

$$\frac{m^2R^4}{M^2r^2} = O(m\dot{r}^2\chi^2)$$

The three-body interaction is of "post-post-Newtonian" order, and this has no observable effect. It can be shown that the formal divergence appearing in this term when the three bodies i, j, and k are aligned is removed when their finite size is taken into account.

The two-body problem has the Lagrangian

$$\mathscr{L} = \frac{1}{2}\left[1 + \frac{3R}{4Mr}\frac{(m_1{}^2 + m_2{}^2)}{m}\right](\dot{r}^2 + r^2\dot{\phi}^2) + \frac{3R\dot{R}^2m}{Mr} + \frac{3Rm}{2M}\left(1 + \frac{m_1{}^2 + m_2{}^2}{4m^2}\right)\frac{\dot{r}^2}{r} \tag{4.1}$$

where $m = m_1 + m_2$, and a center-of-mass frame is taken. [This Lagrangian has an extra term, $\chi r^2\dot{\phi}^2$, over and above that of BB (7.1).] It produces a periastron advance with the same dependence on the orbital elements as general relativity:

$$\Delta\phi = \frac{2\pi}{a(1 - e^2)} \cdot \frac{3Rm}{2M} \left\{ 1 + \frac{3(m_1{}^2 + m_2{}^2)}{4m^2} \right\} \tag{4.2}$$

Using the Mercury perihelion shift, (4.2) gives

$$R/M = 8.5 \times 10^{-28} \text{ m/kg} \tag{4.3}$$

and the measurement of the Kepler period, in conjunction with (4.3), gives

$$\dot{R} = 1.6 \times 10^8 \text{ m/sec} \tag{4.4}$$

Consideration of the restricted three-body problem, consisting of the above system plus the galaxy, produces a mass anisotropy; and a Lagrangian differing only slightly in the coefficients from BB (provided χ is small).

The nearest model we can construct to the case of light deflection is that of a very small mass moving at high velocity near a gravitating body. For a particle moving in from infinity, and with a velocity comparable to $\dot{R}$, the deflection (for small χ) is

$$\Delta\theta \sim \alpha\chi_0 \tag{4.5}$$

where $\alpha \sim 1$ and $\chi_0 \ll 1$ is the perihelion value of the Mach ratio.

5. CONCLUSION

Having constructed a Machian theory of gravity that fulfils in a natural way the equivalence principle, we are ready to discuss inertial frames of reference.

A test particle moves according to a Lagrangian of the form[3]

$$\mathscr{L}_T = A(\mathbf{r}, t) + B_\alpha(\mathbf{r}, t)\dot{r}^\alpha + C_{\alpha\beta}(\mathbf{r}, t)\dot{r}^\alpha\dot{r}^\beta \tag{5.1}$$

where A, B_α, and $C_{\alpha\beta}$ are determined by the positions and velocities of all the other particles. This Lagrangian has been derived in the abstract Euclidean space, introduced at the beginning as a matter of convenience; but the test particle dynamics does not require it. Furthermore, any other labeling of the points of space and time that preserves simultaneity

$$r^\alpha = r^\alpha(\bar{r}^\mu, \bar{t}) \tag{5.2a}$$

$$t = t(\bar{t}) \tag{5.2b}$$

[3] Greek indices range from 1 to 3. Also, $C_{\alpha\beta}$ is positive definite, since any Lagrangian of the form (2.1) reduces to $[C_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta]^{1/2}$ when all other bodies save the test particle are at rest; and this is certainly positive definite.

leaves $\mathscr{L}_T\,dt$ invariant in form, the new coefficients being given by

$$\begin{aligned}
\bar{A} &= A\frac{dt}{d\bar{t}} + B_\alpha \frac{\partial r^\alpha}{\partial \bar{t}} + C_{\alpha\beta}\frac{\partial r^\alpha}{\partial \bar{t}}\frac{\partial r^\beta}{\partial \bar{t}}\frac{d\bar{t}}{dt} \\
\bar{B}_\mu &= B_\alpha \frac{\partial r^\alpha}{\partial \bar{r}^\mu} + 2C_{\alpha\beta}\frac{\partial r^\alpha}{\partial \bar{r}^\mu}\frac{\partial r^\beta}{\partial \bar{t}}\cdot\frac{d\bar{t}}{dt} \\
\bar{C}_{\mu\nu} &= C_{\alpha\beta}\frac{\partial r^\alpha}{\partial \bar{r}^\mu}\frac{\partial r^\beta}{\partial \bar{r}^\nu}\cdot\frac{d\bar{t}}{dt}
\end{aligned} \tag{5.3}$$

We call (5.2) the *restricted covariance group* (Ehlers, 1973). The time variable can still be replaced by $\bar{t}(t)$, say, but if $\bar{t}$ depends on r^α as well, the polynomial structure of (5.1) is lost; thus the group of restricted covariance transformations (5.2) has a dynamical significance and can be distinguished operationally within the general covariant transformations of space-time. If an observer is not aware of the Euclidean reference system of protophysics, he or she will not be able to distinguish between the different frames (5.2); any set of functions (A, B_α, $C_{\alpha\beta}$)—at least for a small domain of space and time—could be attributed to appropriately chosen gravitating bodies.

We now want to use the restricted covariance to find out about inertial frames in the neighborhood of a given test particle $\mathbf{r}_0(t)$. Given a generic Lagrangian $\mathscr{L}(\mathbf{r}, \dot{\mathbf{r}}, t)$, a neighboring body wih an infinitesimal physical displacement (at the same time) $\delta\mathbf{r}(t) = \mathbf{r}(t) - \mathbf{r}_0(t)$, moves according to the Lagrangian

$$\delta\mathscr{L} = \left(\frac{\partial^2\mathscr{L}}{\partial r^\alpha \partial r^\beta}\right)_0 \delta r^\alpha \delta r^\beta + 2\left(\frac{\partial^2\mathscr{L}}{\partial r^\alpha \partial \dot{r}^\beta}\right)_0 \delta r^\alpha \delta \dot{r}^\beta + \left(\frac{\partial^2\mathscr{L}}{\partial \dot{r}^\alpha \partial \dot{r}^\beta}\right)_0 \delta \dot{r}^\alpha \delta \dot{r}^\beta \tag{5.4}$$

(see, e.g., Bliss, 1945); where $(\ )_0$ denotes that the function is computed for $\mathbf{r}_0(t)$, and is therefore a given function of time. In our case, we have

$$\delta\mathscr{L} = C_{\alpha\beta}(t)\delta\dot{r}^\alpha\delta\dot{r}^\beta + D_{\alpha\beta}(t)\delta r^\alpha\delta\dot{r}^\beta + E_{\alpha\beta}(t)\delta r^\alpha \delta r^\beta \tag{5.5}$$

where

$$\begin{aligned}
D_{\alpha\beta} &= 2(B_{\alpha,\beta} + 2C_{\alpha\rho,\beta}\dot{r}_0{}^\rho) \\
E_{\alpha\beta} &= A_{,\alpha\beta} + B_{\rho,\alpha\beta}\dot{r}_0{}^\rho + C_{\rho\sigma,\alpha\beta}\dot{r}_0{}^\rho\dot{r}_0{}^\sigma
\end{aligned}$$

A comma denotes a space derivative. There is an obvious physical interpretation of the three terms in (5.5), corresponding, respectively, to inertia, Coriolis or "magnetic" forces, and tidal forces.

The restricted covariant transformations, when restricted to space [equation (5.2a)], give

$$\delta r^\alpha = P_\beta{}^\alpha \delta \bar{r}^\beta$$

where $P_\beta{}^\alpha(t) = (\partial r^\alpha/\partial \bar{r}^\beta)_0$; upon differentiation we have

$$\delta \dot{r}^\alpha = P_\beta{}^\alpha \delta \dot{\bar{r}}^\beta + \dot{P}_\beta{}^\alpha \delta \bar{r}^\beta$$

One can easily find out the transformation laws for the quantities $C_{\alpha\beta}$, $D_{\alpha\beta}$ and $E_{\alpha\beta}$; all we need here is

$$\bar{D}_{\mu\nu} = C_{\alpha\beta}(P_\nu{}^\alpha \dot{P}_\mu{}^\beta + \dot{P}_\mu{}^\alpha P_\nu{}^\beta) + D_{\alpha\beta} P_\mu{}^\alpha P_\nu{}^\beta$$

We can always set $\bar{D}_{\mu\nu} = 0$ (nine equations) using the nine functions $P_\beta{}^\alpha$. The Coriolis-type terms in (5.5) then disappear, and it is appropriate to say that the corresponding frame is *inertial.* The relevant equations of motion

$$\frac{d}{dt}\,[\bar{C}_{\alpha\beta}(t)\delta\dot{\bar{r}}^\beta] = \bar{E}_{\alpha\beta}(t)\delta\dot{r}^\beta \tag{5.6}$$

are the analog of the equations of geodesic deviation.

There are two reasons why this motion does not agree with ordinary mechanics: There is an anisotropic mass $C_{\alpha\beta}$, and this mass changes with time. If the universe around us were exactly isotropic, $C_{\alpha\beta}$ would be proportional to the unit matrix. Then, as we have done in Section 3, a new time variable $\bar{t}$ can be chosen in such a way that the coefficient of proportionality is equal to, say, 1/2. In the general case, one can still set

$$\bar{C}_{\alpha\beta}(\bar{t}) = \tfrac{1}{2}\delta_{\alpha\beta} + \tfrac{1}{2}\Delta_{\alpha\beta}(\bar{t}), \qquad \Delta_{\alpha\alpha} = 0 \tag{5.7}$$

We still have at our disposal a constant linear substitution in the coordinates $\delta\bar{r}^\alpha$; with it, one can produce now $\Delta_{\alpha\beta}(0) = 0$. In doing this, we have fixed the coordinate axes and introduced a spatial metric for every point of space and for every instant of time. For every phenomenon occurring over a short time scale τ, we have, finally,

$$\bar{C}_{\alpha\beta}(\bar{t}) = \tfrac{1}{2}\delta_{\alpha\beta} + \tfrac{1}{2}\bar{t}\dot{\Delta}_{\alpha\beta}(0) \tag{5.8}$$

The only coordinate freedoms left to us are orthogonal space transformations.

Denoting by δ the amount of anisotropy around us, and by H the expansion rate of the universe, we see that the correction term in (5.8) is of order $H\tau\delta$. There is an upper limit to $\delta(\sim 10^{-3})$ set by the microwave background radiation, and the galaxy produces a δ of order 10^{-7}.

When the gravitational field, described by $E_{\alpha\beta}$, is negligible, we are left with a peculiar statement of the first principle of inertia: Bodies do not move uniformly, but keep constant the momentum

$$P_\alpha = \delta\dot{\bar{r}}_\alpha + t\dot{\Delta}_{\alpha\beta}(0)\delta\dot{\bar{r}}^\beta$$

The set of all free trajectories is *not* invariant under the Galilei group

$$\delta\dot{\bar{r}}^\alpha \to \delta\dot{\bar{r}}^\alpha + v^\alpha$$

the violation is again of order $H\tau\delta$ in dimensionless units, and changes from place to place and from time to time, according to the value of $\dot{\Delta}_{\alpha\beta}(0)$. This is to be regarded as a violation of the strong equivalence principle: Even the simplest dynamical law is not universal.

As already remarked in BB, this preliminary theory cannot be compared quantitatively with experiments until a dynamical description of the measuring instruments—in particular rods and clocks—is provided for. This of course entails reformulating electromagnetism and quantum mechanics in a Machian way, and it is a much harder task. This problem will be subject to severe constraints from the absence of any observed violation of the equivalence principle.

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